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Group Theory of G.Moore

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Contents

1 Concepts of group

1.1 Equivalence relation

Definition 1.1 (Equivalence relation)**.** a,b,c are elements in set X. *Equivalence relation* satisfies

1. $a \sim a$
2. $a \sim b \Rightarrow b \sim a$
3. $a \sim b$ and $b \sim c \Rightarrow a \sim c$

Definition 1.2 (Equivalence class)**.** $\sim$ is an equivalence relation on X. The equivalence class of a is

$$[a] \coloneqq \{x \in X : x \sim a\} \subset X \tag{1.1}$$

a can be any element in $[a]$

Remark. An equivalence relation can decompose X into a union of mutually disjoint subsets. Conversely, a disjoint decomposition can derive an equivalence relation.

1.2 Group

Definition 1.3 (Group)**.** A *group* is a quartet $(G, \mathbf{m}, \mathbf{I}, e)$

1. G is a set
2. $\mathbf{m}$ is a map (*group multiplication map*): $G \times G \to G$
3. $\mathbf{I}$ is a map (*inverse map*): $G \to G$
4. $e \in G$ is a distinguished element of G called the *identity element*.

Definition 1.4 (Direct product)**.** Two groups G_1, G_2, direct product of G_1, G_2 is a set $G_1 \times G_2$ with product:
$\mathbf{m}_{G_1 \times G_2}\left(\left(g_1, g_2\right), \left(g_1', g_2'\right)\right) = \left(\mathbf{m}_{G_1}\left(g_1, g_1'\right), \mathbf{m}_{G_2}\left(g_2, g_2'\right)\right)$

Definition 1.5 (Order)**.** *Order* of a group G is the number of elements in G. It is also the cardinality of G as a set.

Definition 1.6 (Abelian)**.** A group G is an Anelian group if any pair of elements in G is commute i.e. $a \cdot b = b \cdot a$

Definition 1.7 (Center)**.** The center $Z(G)$ of the group G is a set of elements $z \in G$ that commute with all elements in G.

$$Z(G) \coloneqq \{z \in G | zg = gz \ \forall g \in G\}$$

Example 1.1. The General Linear Group and Linear Transformation Group

$$GL(n,\kappa) \coloneqq \{A|A = n \times n \text{ invertable matrix over} \kappa\}$$

$$GL(V) \coloneqq \{\text{invertible linear transformations from V to itself }\}$$

Example 1.2. Some Standard Matrix Groups

1. The special linear group

$$SL(n,\kappa) \coloneqq \{A \in GL(n,\kappa) : \det A = 1\}$$

2. The orthogonal group

$$O(n,\kappa) \coloneqq \left\{A \in GL(n,\kappa) : AA^{tr} = 1\right\}$$

3. The special orthogonal group

$$SO(n,\kappa) \coloneqq \{A \in O(n,\kappa) : \det A = 1\}$$

4. The unitary group

$$U(n) \coloneqq \left\{A \in GL(n,\mathbb{C}) : AA^{\dagger} = 1\right\}$$

5. The special unitary group

$$SU(n) \coloneqq \{A \in U(n) : \det A = 1\}$$

Example 1.3. Some groups defined by bilinear forms

b is a bilinear form map i.e.

$$b(x,y) = x^{tr} b y = c \quad x,y \in V \ , \quad c \in \mathbb{R} \ or \ \mathbb{C}, \quad b \in M_n(\kappa)$$

$$b(\alpha x, \beta y) = \alpha\beta c, \quad b(x_1+x_2, y_1+y_2) = b(x_1,y_1) + b(x_2,y_1) + b(x_1,y_2) + b(x_2,y_2)$$

Define an automorphism group of the bilinear form:

$$\mathrm{Aut}(b) \coloneqq \{T \in M_n(\kappa) | b(Tx,Ty) = b(x,y) \quad \forall x,y\}$$

$$\mathrm{Aut}(b) \coloneqq \{T \in M_n(\kappa) | T^{tr} bT = b\}$$

If we want to derive T is invertible from the definition , we should let b be invertible. Taking the transpose of this equation suggests that $b^{tr} = \lambda b$ and $\lambda = \pm 1$.

So there are symmetric and antisymmetric forms of b. That leads to two kinds of automorphism groups e.g. *symplectic matrix* and *Lorentz group*. *Symplectic matrix* is the transform matrix of the canonical coordinates and momenta called canonical transformations.

Example 1.4 (Lorentz group).

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1_{n\times n} \end{pmatrix}$$

the set of matrices $O(1,n) := \{A|A^{tr}\eta A = \eta\}$ is known as the Lorentz group of Minkowski spacetime in 1+n dimensions.

Example 1.5 (Symplectic group).

$$J = \begin{pmatrix} 0 & 1_{n\times n} \\ -1_{n\times n} & 0 \end{pmatrix}$$

$$Sp(2n,\kappa) := \{A \in GL(2n,\kappa)|A^{tr}JA = J\}$$

Example 1.6. Loop group

$$LG := \{f \text{ is a function}|f(S^1 \to G)\}$$

Example 1.7. Permutation Groups S_X A permutation of a set X is a one-one invertible transformation $\phi : X \to X$.

Definition 1.8 (Power Set)**.** Power Set is all the subset of a set.

Definition 1.9. Some property of matrix

1. Symplectic $A^{tr}JA = J$
2. Complex $A^{-1}JA = J$
3. Orthogonal $A^{-1} = A^{tr}$

Any two of them imply the third.

2 Homomorphism and Isomorphism

Definition 2.1 (Homomorphism and Isomorphism)**.** Homomorphism is a mapping φ between two groups that preserves the group law i.e.

$$\varphi(\mathbf{m}(g_1, g_2)) = \mathbf{m}'(\varphi(g_1), \varphi(g_2))$$

1. If $\varphi(g) = 1_{G'}$ implies that $g = 1_G$ then φ is called *injective* or *into*.
2. If $\forall g' \in G'$ $\exists g \in G$ s.t. $\varphi(g) = g'$ then φ is called *surjective* or *onto*.
3. If φ is both *into* and *onto*, then φ is called *isomorphism*
4. If φ is *isomorphism.* G and G' are groups with the same set and the same multiplication law. φ will be called *automorphism*

Definition 2.2 (Commutative diagrams)**.** A diagram like the one below is *commutes* iff we follow the arrows around two paths with the same beginning and finish point, composing the mappings on these arrows and getting two equal mappings.

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ C & \xrightarrow{\eta} & D \end{array}$$

Definition 2.3 (Representation of the group)**.** The (finite-dimensional) representation of a group G is a vector space V together with a homomorphism $\varphi : G \to GL(V)$. V is called the carrier space.

Example 2.1. $SO(2)$ is isomorphism to $U(1)$

Definition 2.4 (Kernel and image)**.** Kernel and image are two nature subgroups:

1. The *kernel* of homomorphism μ is

$$ker\mu := \{g \in G | \mu(g) = 1_{G'}\}$$

2. The *image* of μ is

$$im\mu := \mu(G) \subset G'$$

Example 2.2. There is an important *Homomorphism* from $SU(2)$ to $SO(3)$.

for any $u \in SU(2)$ there is a homomorphism $R(u) \in SO(3)$ s.t. $\forall \vec{x} \in \mathbb{R}^3$

$$u\vec{x} \cdot \vec{\sigma} u^{-1} = (R(u)\vec{x}) \cdot \vec{\sigma}$$

In fact, we can define an isomorphism h

$$h : \mathbb{R}^3 \to \mathcal{H}$$

$$h(\vec{x}) := \vec{x} \cdot \vec{\sigma}$$

We can also define a conjugate of $u \in SU(2)$

$$C_u : \mathcal{H}_2 \to \mathcal{H}_2$$

$$C_u(m) := umu^{-1}$$

Therefore we can define $R(u)$ with a diagram

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{R(u)} & \mathbb{R}^3 \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ \mathcal{H}_2 & \xrightarrow{C_u} & \mathcal{H}_2 \end{array}$$

3 Group Actions On Sets

3.1 Conceptions

Definition 3.1 (permutation)**.** A permutation of X is an into and onto mapping from X to X, forming a set S_X, and it forms a group under composition.

Definition 3.2 (left G-action)**.** A left G-action on a set X is a map $\phi := G \times X \to X$ compatible with

$$\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$$

We also impose the condition

$$\phi(1_G, x) = x \ \forall x \in X$$

This map is a homomorphism from G to S_X

Definition 3.3 (G-set)**.** If X has a group action by a group G we say that X is a G-set.

Definition 3.4 (Orbit)**.** If G acts on a set X we can define an equivalence relation on X. $x_1.x_2 \in X$ are equivalent $x_1 \sim x_2$ if there is some $g \in G \ s.t. \ \phi(g, x_1) = x_2$.

$$O_G(x) = \{y : \exists g \ s.t. \ y = g \cdot x\}$$

The set of orbits is denoted X/G.

Remark. If X, G are topological spaces, with the G action on X continuous the X/G carries a natural topology. (refer to G. Moore Ch1 Page33)

Definition 3.5 (Equivariant map)**.** If X, X' are two G-spaces. We have an equivariant map $f : X \to X'$

$$f(g \cdot x) = g \cdot f(x)$$

or

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow{\scriptstyle \Phi(g)} & & \downarrow{\scriptstyle \Phi'(g)} \\ X & \xrightarrow{f} & X' \end{array}$$

Remark. Isomorphism is a bijective equivariant map.

Definition 3.6 (Induce Group Actions)**.** Suppose X and Y are two sets and $\mathcal{F}[X \to Y]$ is the set of functions from X to Y. If there is a left G-action on X defined by $\phi : G \times X \to X$. There will automatically a G-action $\tilde{\phi} : G \times \mathcal{F} \to \mathcal{F}$ on $\mathcal{F}[X \to Y]$.

$$\tilde{\phi}(g, F) := F(\phi(g^{-1}, x))$$

Remark. Note the inverse of g on the RHS. It guarantees the properties:

$$\begin{aligned}
\tilde{\phi}(g_1, \tilde{\phi}(g_2, F))(x) &= \tilde{\phi}(g_2, F)(\phi(g_1^{-1}, x)) \\
&= F(\phi(g_2^{-1}, \phi(g_1^{-1}, x))) \\
&= F(\phi(g_2^{-1} g_1^{-1}, x)) \\
&= F(\phi((g_1 g_2)^{-1}, x)) \\
&= \tilde{\phi}(g_1 g_2, F)(x)
\end{aligned} \tag{3.1}$$

3.2 More about group actions and orbits

Definition 3.7 (left and right action)**.**

1. left-action:
$$\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$$
2. right-action:
$$\phi(g_1, \phi(g_2, x)) = \phi(g_2 g_1, x)$$

Remark.

1. left-action can be written as:
$$g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$$
2. right-action can be written as:
$$(x \cdot g_2) \cdot g_1 = x \cdot (g_2 \cdot g_1)$$
3. If ϕ is a left-action the $\tilde{\phi}(g, x) := \phi(g^{-1}, x)$ is a right-action.
4. A given set X can admit more than one action by the same group G. For example, if $X = G$:
$$L(g, g') = gg' \tag{3.2}$$
$$\tilde{L}(g, g') = g^{-1} g' \tag{3.3}$$
$$R(g, g') = g' g \tag{3.4}$$
$$\tilde{R}(g, g') = g' g^{-1} \tag{3.5}$$
$$C(g, g') = g^{-1} g' g \tag{3.6}$$
$$\tilde{C}(g, g') = g g' g^{-1} \tag{3.7}$$

Definition 3.8.

1. A group action is *ineffective* if there is some $g \in G$ with $g \neq 1$ *s.t.* $g \cdot x = x \ \forall x \in X$. We can regard group action as a homomorphism $G \to S_X$, then the action is effective means the homomorphism is injective. The set of g that act ineffectively is the ker of the homomorphism and they form a normal subgroup of G.

2. *transitive*: for any pair $x, y \in X$ there is some g with $y = g \cdot x$.

3. *free*: for any $g \neq 1$ for every x, we have $g \cdot x \neq x$.

In summary:
Effective: If $g \neq 1$, $\phi_g \neq 1$
Transitive: You can always find g that links any two pairs.
Free: All g except $g = 1$, change every element in X.

Definition 3.9 (isotropy group (stabilizer group))**.** Given a point $x \in X$

$$Stab_G(x) := \{g \in G : g \cdot x = x\} \subset G$$

is called the isotropy group at x. It is also called the stabilizer group of x denoted G^x. A group action is free iff for every $x \in X$ so that G^x is the trivial subgroup $\{1_G\}$

Definition 3.10 (fixed point)**.** A point $x \in X$ is a fixed point of the G-action if there exists some elements $g \in G$ with $g \neq 1$ s.t. $g \cdot x = x$. That means $Stab_G(x)$ is nontrivial. We don't need $Stab_G(x) = G$.

Definition 3.11 (fixed point set)**.** The fixed point set of g is the set

$$Fix_X(g) := \{x \in X : g \cdot x = x\} \subset X$$

denoted by X^g. If the group action is free, the set $Fix_X(g)$ is empty fro all $g \neq 1$.

Remark. The group action is restricted to a transitive group action on any orbit.

If x,y are in the same orbit then the isotropy groups G^x, G^y are conjugate subgroups in G.

Definition 3.12. If G acts on X. A stratum is a set of G-orbits s.t. the conjugacy class of the stabilizer groups is the same. The set of strata is denoted $X \| G$

Definition 3.13 (Derangements)**.** The permutation without fixed points is called a derangement. the number of derangements in S_n is

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

If we have an infinitely large set and the probability of choosing a derangement is $e^{-1} = 0.3678 \cdots$

Definition 3.14 (induced group action)**.** Let X be a G-set and let Y be any set. Given the function $\Psi \in Map(X, Y)$ and $g \in G$, we need to produce a new function $\phi(g, \Psi) \in Map(X, Y)$

1. If G is a left-action on X then:

$$\phi(g, \Psi)(x) := \Psi(g \cdot x) \qquad \text{right action on } Map(X, Y)$$

2. If G is a left-action on X then:

$$\phi(g,\Psi)(x) := \Psi(g^{-1}\cdot x) \qquad \text{left action on } Map(X,Y)$$

3. If G is a right-action on X then:

$$\phi(g,\Psi)(x) := \Psi(x\cdot g) \qquad \text{left action on } Map(X,Y)$$

4. If G is a right-action on X then:

$$\phi(g,\Psi)(x) := \Psi(x\cdot g^{-1}) \qquad \text{right action on } Map(X,Y)$$

Definition 3.15 (Poincaré transformation)**.** Suppose $X = \mathbb{M}^{1,d-1}$ is d-dimensional Minkowski space time, G is the Poincaré group and $Y = \mathbb{R}$. Given one scalar field Ψ and a Poincaré transformation $g^{-1}\cdot x = \lambda x + v$. We have

$$(g\cdot\Psi)(x) = \Psi(\lambda x + v)$$

Definition 3.16 (general action)**.** If X is a G_1-set and Y is a G_2-set. We have a natural $G_1 \times G_2$-action on Map(X,Y) defined by

$$\phi((g_1,g_2),\Psi)(x) := g_2\cdot(\Psi(g_1^{-1}\times x))$$

Example 3.1. Let $X = \mathbb{M}^{1,d-1}$, G to be the Poincaré group and $Y = V$ is the finite-dimensional representation of the Poincaré group. We denoted the action of $g \in G$ on V by $\rho(g)$. Then we have

$$\phi(g,\Psi)(x) = \rho(g)\Psi(g^{-1}x)$$

Theorem 3.1 (The Stabilizer-Orbit Theorem)**.** There is a natural isomorphism of G-sets

$$\psi : Orb_G(x) \to G/G^x$$

Proof. Suppose y is in a G-orbit of x. Then $\exists g\ s.t.\ y = g\cdot x$. Define

$$\psi(y) = g\cdot G^x$$

It is G-equivariant:

$$\psi(gy) = g\psi(y)$$

and invertible:

$$\psi^{-1}(gG^x) = g\cdot x$$

So it is an isomorphism. □

Corollary: If G acts transitively on X then the isotropy groups G^x for all the points $x \in X$ are conjugate subgroups of G

Example 3.2. This theorem connected algebraic notions of subgroups and cosets to the geometric notions of orbits and fixed points.

If G,H are topological groups then, sometimes G/H are beautifully symmetric topologihcal spaces, if G,H are Lie groups then, sometimes, G/H are symmetric manifolds.

One example of homogeneous spaces is in describing the vacua of a scalar field theory with a global symmetry.

Suppose ϕ is a scalar field on d-dimensional Minkowski space,$\mathbb{M}^{1,d-1}$, valued in some representation space V of a compact Lie group G.Suppose $U(\phi)$ is a G-invariant potential energy.

Theorem 3.2. As a claim, there is a theorem:

Consider a group $\mathbb{Z}_p$ with p as a prime number. We can prove that any element $g \neq np$ is a generator of the group.

Corollary: the Orbit of $\mathbb{Z}_p$ can only be a single point or p distinct points.

Theorem 3.3 (Burnside lemma)**.** Suppose a finite group G acts on a finite set X as a transformation group

The number of the orbits has a relation with the averaged number of the fixed points.

$$|\{orbits\}| = \frac{1}{|G|} \sum_g |X^g|$$

In fact, $\sum_g |X^g| = \sum_x |G^x|$

Theorem 3.4 (Jordan's theorem)**.** Suppose G is finite and acts transitively on a finite set X with more than one point. Show that there is an element $g \in G$ with no fixed points on X.

Hint: The number of orbit is one and apply Burnside lemma.

Example 3.3.

1. For $X = \{1, \cdots, n\}$ and the actions are S_n. The action is effective and transtive but not free. The fixed point of any $j \in X$ is just the permutations of everything else.

   $$S_X^j \cong S_n$$

2. $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication. If we act with a matrix on a column vector we get a left action. If we act on a row vector we get a right action.

   Remark: it is not transitive because it can only map $\vec{x} = 0$ to 0. It is also not free, $\vec{0}$ is a fixed point. There are two orbits.

3. If we change the restrict from GL to SO and consider n=2. There will be infinitely many orbits of $SO(2)$ and they are distinguished by the invariant value of $x^2 + y^2$. From the viewpoint of topology, there are two distinct "kinds" of orbits, give two strata.

4. $$Stab_{SO(3)}(\hat{n}) \cong SO(2)$$

For $\hat{n} \in S^2$

$$S^2 \cong SO(3)/SO(2)_{\hat{n}}$$

Fixing any $\hat{n}$. There is a map

$$\pi_{\hat{n}} : SO(3) \to S^2$$

$$\pi_{\hat{n}}(R) := R \cdot \hat{n} \in S^2$$

The inverse will be a set:

$$\pi_{\hat{n}}^{-1}(\hat{k}) := \{R | R\hat{n} = \hat{k}\} \subset SO(3)$$

This set will be in 1-1 correspondence with elements of $SO(2)$. $\hat{k} = R_1\hat{n} = R_2\hat{n}$ and $R_1^{-1}R_2\hat{n} = \hat{n}$. Then $R_1^{-1}R_2 = R_0 \in Stab_{SO(3)}(\hat{n}) \cong SO(3)$

We can define subgroups of $SO(3)$

$$R_{12}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{23}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

In fact, The general element of $SO(3)$ can be written as

$$R = R_{12}(\phi)R_{23}(\theta)R_{12}(\psi)$$

If we set $\theta = 0$ then we get a one-dimensional subspace of SO(3): $R_{12}(\phi + \psi)$. This explains why the generic matrix has a unique Euler angle presentation but the manifold $SO(3)$ is not the same as $S^2 \times S^1$

5. $GL(2, \mathbb{C})$ and $SU(2)$ act on $\mathbb{CP}^1$

   $\mathbb{CP}^1$ is the equivalence classes of points $(z_1, z_2) \in \mathbb{C}^2 - \{0\}$ with quivalence relation $(z_1', z_2') \sim (\lambda z_1, \lambda z_2)$ denoted by $[z_1 : z_2]$

   We can also regard it as the space of a states of a single Qbit with $|z_1|^2 + |z_2|^2 = 1$.

$$\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

   Obviously, $GL(2)$ action is transitive and therefore we have an identification:

$$GL(2, \mathbb{C})/Stab_{GL(2,\mathbb{C})}(p) \cong \mathbb{CP}^1 \qquad p \in \mathbb{CP}^1$$

For $SU(2)$, it is still transitive. The stabilizer of $[1:0]$ is isomorphic to $U(1)$. Therefore, there is also an identification

$$\mathbb{CP}^1 \cong SU(2)/U(1)$$

Hence, there is a nature map

$$\pi : SU(2) \to \mathbb{CP}^1$$

We can use Riemann sphere to identify $\mathbb{CP}^1$ by stereographic projection. We can define a map

$$\phi_N : \mathcal{U}_N \to \mathbb{C} \qquad \phi_N([z_1 : z_2]) := z_S := z_2/z_1$$

where $z_1 \neq 0$. In terms of the identification $S^2 \cong \mathbb{CP}^1$

$$z_N = \frac{x_1 + \mathrm{i}x_2}{1 - x_3}$$

or

$$z_S = \frac{x_1 - \mathrm{i}x_2}{1 + x_3}$$

Since $S^3 \cong SU(2)$, we have a nature continuous map:

$$\pi : S^3 \to S^2$$

whose fibers are copies of S^1. This is a famous map known as the *Hopf map*. This is very closely related to the map $\pi : SO(3) \to S^2$.

4 The Symmetric Group

We shall soon see, that all finite groups are isomorphic to subgroups of the symmetric group.

Definition 4.1 (left and right operations)**.**

$$(\phi_1 \cdot_L \phi_2)(i) := \phi_2(\phi_1(i))$$

$$(\phi_1 \cdot_R \phi_2)(i) := \phi_1(\phi_2(i))$$

$\cdot_L$ means we read the operations from left to right and apply the left permutation first, then the right.

Definition 4.2 (permutation)**.** We can write a permutation symbolically as

$$\phi = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$$

That means $\phi(1) = p_1, \phi(2) = p_2. \cdots, \phi(n) = p_n$, This is equal to

$$\phi = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ p_{a_1} & p_{a_2} & \cdots & p_{a_n} \end{pmatrix}$$

Remark.

$$\phi_1 \cdot_L \phi_2 = (\phi_1^{-1} \cdot_R \phi_2^{-1})^{-1}$$

Definition 4.3 (Canonical Permutation Representation)**.** Consider the n dimension vector space with basis vectors $\vec{e}_1, \cdots, \vec{e}_n$. The symmetric group permutes these vectors in an obvious way:

$$T(\phi) : \vec{e}_i \to \vec{e}_{\phi(i)}$$

$$T(\phi) : \sum_{i=1}^{n} x_i e_i \to \sum_{i=1}^{n} x_i e_{\phi(i)} = \sum_{i=1}^{n} x_{\phi^{-1}}(i) e_i$$

Remark.

$$T(\phi_1) \circ T(\phi_2) = T(\phi_1 \circ \phi_2) = T(\phi_1 \cdot_R \phi_2)$$

$$(T(\phi)^{-1} A T(\phi))_{i,j} = A_{\phi(i),\phi(j)}$$

Definition 4.4 (Signed Permutation Matrices)**.** Signed permutation matrices are matrices *s.t.* in each row and column there is only one nonzero matrix element and the nonzero element can be eight $+1$ or -1. The set of $n \times n$ signed permutation matrices form a group. We denote it by $W(B_n)$

Theorem 4.1 (Cayley's Theorem)**.** Any finite group is isomorphic to a subgroup of permutation group S_N for some N

Definition 4.5 (Reducible)**.** In general, if we have a representation of a group $T : G \to GL(V)$and a nontrivial subspace $W \subset V$ such that $T(g)$ takes vectors in W to vectors in W for all $g \in G$, we say that the representation is reducible. Representation Theory

Definition 4.6 (Cyclic Permutation)**.** The cyclic permutations of length is

$$a_1 \to a_2 \to a_3 \to \cdots \to a_l$$

denoted as

$$\phi = (a_1 a_2 \cdots a_l) = (a_2 a_3 \cdots a_l a_1)$$

Remark. Usually, the multiply is $\cdot_R$

Definition 4.7 (Cycle Decomposition)**.** Any permutation $\sigma \in S_n$ can be uniquely written as a product of disjoint cycles. This is called the cycle decomposition of σ.

Definition 4.8 (Transposition)**.** A transposition is a permutation of the form (ij). Transpositions obey the following identities:

$$(ij)(jk)(ij) = (jk)(ij)(jk)$$

$$(ij)^2 = 1$$

$$(ij)(kl) = (kl)(ij) \qquad \{i, j\} \cap \{i, j\} = \emptyset$$

Remark. Every element of Sn can be written as a product of transpositions. We say that the transpositions *generate* the permutation group.

We might call the products a "word" and the "letters" are the transpositions $(1,k)(1,k-1)\cdots(1,4)(1,3)(1,2) = (1,2,3,4,\cdots,k)$

Definition 4.9 (alternating group)**.** The alternating group A_n is the subgroup of S_n of even permutations.

Remark. The order of A_n is $\frac{n!}{2}$.

Example 4.1 (Two kinds of generators sets)**.** Two smaller sets of generators of S_n

1. $\sigma_i := (i, i+1), 1 \leq i \leq n-1$
2. (12) and $(12\cdots n)$

5 Cosets and conjugacy

5.1 Lagrange Theorem

Definition 5.1 (Left-coset)**.** Let $H \subseteq G$ be a subgroup. The set

$$gH \equiv \{gh|h \in H\} \subset G$$

is called a left-coset of H.

Remark. Two left cosets are either identical or disjoint. That means the cosets define an equivalence relation by saying $g_1 \sim g_2$ is the is an $h \in H$ $s.t.$ $g_1 = g_2h$

Theorem 5.1 (Lagrange Theorem)**.** If H is a subgroup of a finite group G then the order of H divides the order of G:

$$|G|/|H| \in \mathbb{Z}_+$$

Lemma 5.1.1. Any finite group of prime order p is isomorphic to $\mu_p \cong \mathbb{Z}_p$ with only two subgroups: the trical group and itself.

Definition 5.2 (homogeneous space(set of left cosets))**.** If G is any group and H any subgroup then the set of left cosets of H in G is denoted G/H. It is the set of orbits under the right H action on G. A set of the form G/H is also referred to as a homogeneous space. The cardinality of this set is the index of H in G, and denoted $[G : H]$.

Remark. A converse to Lagrange's theorem is not true. If $n||G|$ (a divides b), There might be no subgroup of G of order n.

Theorem 5.2 (Sylow's first theorem)**.** Suppose p is prime and p^k divides $|G|$ for a non-negative integer k. Then there is a subgroup $H \subset G$ of order p^k. Centralizer Subgroups And Counting Conjugacy Classes

Definition 5.3 (Order of the element)**.** An element $g \in G$ has order n if n is the smallest natural number such that $g^n = 1$

5.2 Conjugacy

Definition 5.4 (Conjugacy (similarity of matrices))**.**

1. A group element h is conjugate to h' means $\exists h \in G \quad h' = ghg^{-1}$
2. Conjugacy defines an equivalence relation and the conjugacy class of h is the equivalence class under this relation:
$$C(h) := \{ghg^{-1} : h \in H\}$$
3. Let $H \subseteq G, K \subseteq G$ be two subgroups. We say "H is conjugate to K" if $\exists g \in G \; s.t.$
$$K = gHg^{-1} := \{ghg^{-1} : h \in H\}$$
4. Two homomorphism $\varphi_i : G_1 \to G_2$ are conjugate if $\exists g_2 \in G_2 \; s.t.$
$$\varphi_2(g_1) = g_2\varphi_1(g_1)g_2^{-1} \qquad \forall g_1 \in G_1$$
5. A matrix representation of G is a homomorphism
$$\varphi : G \to GL(n, \kappa)$$
If two representations are conjugate, we say they are $equivalent representations$.

Remark. Put differently, two representations $T_1 : G \to GL(V_1)$, $T_2 : G \to GL(V_2)$ are said to be equivalent if there is an isomorphism map $S : V_1 \to V_2$. (We will discuss "intertwiners" in Representation Theory)

Example 5.1. All cyclic permutations in S_n are conjugate.

Definition 5.5 (Maximal torus)**.** For compact Lie group G. any Abelian subgroup of G is isomorphic to $U(1)^r$. r is called the *rank*. Any two maximal Abelian subgroups are conjugate groups. Such a maximal Abelian subgroup is the maximal torus.

Definition 5.6 (Class function)**.** A class function on a group is a function f on G such that
$$f(g_0 g g_0^{-1}) = f(g) \qquad \forall g_0, g \in G$$

Example 5.2. If φ is a matrix representation then
$$\chi_\varphi(g) := \mathrm{Tr}\, \varphi(g)$$
is a example of a class function. We call it the character of the representation.

Another class function is the characteristic polynomial:
$$p_A(x) := \det(xI - A)$$
($p_A(x)$ is the value of the function.)

5.3 Normal Subgroups And Quotient Groups

Definition 5.7 (Normal subgroup)**.** A subgroup $N \subseteq G$ is called *normal* subgroup (*invariant* subgroup) if

$$gNg^{-1} = N \qquad \forall g \in G$$

denoted as $N \triangleleft G$

Theorem 5.3 (Quotient group)**.** If $N \subset G$ is a normal subgroup then the set of left cosets $G/N = \{gN | g \in G\}$ has a natural group structure:

$$(g_1 N)(g_2 N) := (g_1 \cdot g_2)N$$

We call it a Quotient group

Remark.

1. All subgroups N of Abelian groups A are normal, and the quotient group A/N is Abelian.
2. If we have the representation $T : G \to GL(n, \kappa)$ of G and if $H \subset G$ is a subgroup then we can restrict T to H to get a representation of H. But the quotient of G: Q can not get a representation from T.

Example 5.3 (Cyclic group)**.**

$$\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n \text{ or } \mu_n$$

Definition 5.8 (simple group)**.** A group with only $\{1\}$ and G being its normal subgroup, i.e. with no nontrivial normal subgroup, is called a *simple group*.

Theorem 5.4 (Cancellation Theorem)**.** Suppose we have a chain of normal subgroups:

$$K \triangleleft N \triangleleft G$$

and $K \triangleleft G$

1. $$N/K \triangleleft G/K$$
2. There is an isomorphism of groups

$$G/N \cong (G/K)/(N/K)$$

Definition 5.9 (p-Sylow subgroup)**.** Consider prime p, if we take the largest prime power p^k dividing $|G|$, the a subgroup of order p^k is called a p-Sylow subgroup.

Theorem 5.5 (Sylow's second theorem)**.** All the p-Sylow subgroups are conjugate.

Remark. Sylow's third theorem is about the number of p-Sylow's subgroups.

Definition 5.10 ($PSU(N)$)**.** The center of $U(N)$ consists of matrices proportional to the unit matrix.

The center of $SU(N)$ should also be diagonal and $zI_{N\times N}$ in $SU(N)$ means $z^N = 1$. So $Z(SU(N)) \cong \mu_N \cong \mathbb{Z}_N$.

$$PSU(N) := SU(N)/\mathbb{Z}_N$$

$$PSU(N) \cong U(N)/Z(U(N)) \cong U(N)/U(1)$$

Theorem 5.6. If the center $Z(G)$ *s.t.* $G/Z(G)$ is cyclic then G is Abelian.

Definition 5.11 (Group Commutator)**.** If g_1, g_2 are elements of group G the the group commutator is defined to be

$$[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$$

Definition 5.12 (Commutator subgroup)**.** Commutator subgroup of G is denoted by $[G, G]$ or G' is the subgroup generated by words in all group commutator $g_1 g_2 g_1^{-1} g_2^{-1}$

Remark. $[G, G]$ is a normal subgroup of G $(g_0 [g_1, g_2] g_0' = [g_1', g_2'])$

Definition 5.13 (Abelianization)**.** $G/[G, G]$ is abelian. This is called the *abelianization* of G.

Definition 5.14 (Perfect group)**.** A perfect group is a group which is equal to its commutator subgroup.

Remark. A nonabelian simple group must be perfect.

Example 5.4. The commutator subgroup of S_n is A_n

Definition 5.15 (The Normalizer Subgroup)**.** Consider $H \subset G$ is a subgroup of G. The *normalizer* of H within G is the largest subgroup N of G such that H is a normal subgroup of N.

$$N_G(H) := \{g \in G | gHg^{-1} = H\}$$

is a subgroup of G and H is a normal subgroup of $N_G(H)$.

Example 5.5 (Quotient Groups of SU(2))**.** $D \subset SU(2)$ is subgroup of diagonal matrices.

$$SU(2) := \begin{pmatrix} z & -w^* \\ w & z^* \end{pmatrix} \qquad |z|^2 + |w|^2 = 1$$

$$D := \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix} \qquad |a|^2 = 1$$

Note that $D \cong U(1)$.

Then

$$N_{SU(2)}(D) := \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \; or \; \begin{pmatrix} 0 & -z^{-1} \\ z & 0 \end{pmatrix} \quad z \; is \; phase$$

$$N_{SU(2)}(D)/D \cong \mathbb{Z}_2$$

Definition 5.16 (Weyl Group)**.** T is a maximal torus of Lie group G. The Weyl group of G is :

$$W(G) := N_G(T)/T$$

Remark.

$$W(SU(n)) := N_{SU(2)}(D)/D \cong S_n$$

Theorem 5.7. *$N \triangleleft G$. Define a homomorphism $\varphi : G \to \tilde{G}$. $\varphi(N) \subset \tilde{G}$ is a subgroup. It is a normal subgroup iff φ is surjective.*

5.4 Short Exact Sequences

Definition 5.17 (Exact sequence)**.** Suppose we have three groups and two homomorphism, an *exact sequence* is

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3$$

$$\boxed{\boxed{\text{the sequence is exact at } G_2 \text{ iff } \operatorname{im} f_1 = \ker f_2}}$$

Definition 5.18 (Short exact sequence)**.** A short exact sequence is a sequence of the form

$$1 \to G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to 1$$

exact at G_1, G_2, G_3

Remark.

1. Exactness at G_1: f_1 is an injective homomorphism.
2. Exactness at G_2: $\operatorname{im} f_1 = \ker f_2$
3. Exactness at G_3: f_1 is a surjective homomorphism.

Theorem 5.8. *Suppose we have a group homomorphism $\mu : G \to G'$. We have a natural short exact sequence:*

$$1 \to \ker(\mu) \to G \xrightarrow{\mu} \operatorname{im}(\mu) \to 1$$

$\ker(\mu) \subset G$ is a normal subgroup of G

Theorem 5.9.

$$\boxed{\mu(G) \cong G/\ker(\mu)}$$

Theorem 5.10. *With a short exact sequence*

$$1 \to N \to G \to Q \to 1$$

N is automatically isomorphic to a normal subgroup of G. Q is isomorphic to G/N. Then we say "G is an extension of Q by N" (or G is an extension of N by Q)

Remark. In quantum mechanics physical states are actually represented by "rays" in Hilbert space (one-dimensional subspaces of Hilbert space or orthogonal projection operators of rank one—— density matrices).

When comparing symmetries of quantum systems with their classical counterparts, group extensions play an important role. Group Extensions and Group Cohomology

Example 5.6. Consider a homomorphism $\pi : \mu_4 \to \mu_2$ given by $\pi(g) = g^2$. The kernel is μ_2. So

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1$$

Furthermore

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 1$$

p is any prime.

Example 5.7 (Finite Heisenberg Groups)**.** Let P, Q be "clock" and "shift" matrices. $\omega = \exp{[2\pi i/N]}$

$$P_{i,j} = \delta_{i=j+1 mod N}$$

$$Q_{i,j} = \delta_{i,j}\omega^j$$

Note that $P^N = Q^N = 1$ and $QP = \omega PQ$

These forms a group with elements:

$$\omega^a P^b Q^c$$

$$(\omega^{a_1} P^{b_1} Q^{c_1}) \cdot (\omega^{a_2} P^{b_2} Q^{c_2}) = \omega^{a_3} P^{b_3} Q^{c_3}$$

$$a_3 = a_1 + a_2 + c_1 b_2$$

$$b_3 = b_1 + b_2$$

$$c_3 = c_1 + c_2$$

This group is called a finite Heisenberg group and is denoted by Heis_N.8 It is an extension

$$1 \to \mathbb{Z}_N \to \mathrm{Heis}_N \xrightarrow{\pi} \mathbb{Z}_N \times \mathbb{Z}_N \to 1$$

$$\pi(\omega^a P^b Q^c) = (b \mod N, c \mod N)$$

5.5 Conjugacy Classes In S_n

If $(i_1 i_2 \cdots i_k)$ is a cycle of length k then $g(i_1 i_2 \cdots i_k)g^{-1}$ is a cycle of length k. It is the cycle where we replace $i_1 i_2 \cdots i_k$ by their images under g. i.e. $g(i_a) = j_a$, then $g(i_1 i_2 \cdots i_k)g^{-1} = (i_1 j_2 \cdots j_k)$

Therefore, any two cycles of length k are conjugate.

It is easy to prove that any element in S_n can be written as a product of disjoint cycles. Therefore, the conjugacy classed in S_n is labeled by a series of nonnegative integers, denoted l_j, which are the number of distinct cycles of length j.

Definition 5.19 (Partition of n)**.** A decomposition of n into a sum of nonnegative integers is called a partition of n.

The conjugacy classes of S_n are in 1-1 correspondence with the partitions of n.

Definition 5.20. The number of distinct partitions of n is called the partition function of n and denoted $p(n)$

Remark. This is different from the partition function in field theory!

Definition 5.21 (Order of the conjugacy class)**.**

$$|C(g)| = \frac{n!}{\Pi_{j=1}^{n} j^{l_j} l_j!}$$

Definition 5.22 (Sign of the conjugacy class)**.** The sign of a conjugacy class is $\epsilon(g) = (-1)^{n+\Sigma_j l_j}$

Proof. Suppose the partition is $(\lambda_1, \lambda_2, \cdots, \lambda_k)$. Here k is the total number of disjoint cycles $k = \sum_j l_j$.

Rewrite the cycle with transpositions have the number of transpositions as:

$$(\lambda_1 - 1) + (\lambda_2 - 1) + \cdots + (\lambda_k - 1) = \sum_i^k \lambda_i - k = n - k$$

The sign of a cycle is $sgn(\sigma) = -1$ if σ can be written as an odd number of transposition and v/v. □

Example 5.8 (Harmonic Oscillators)**.**

$$H = \frac{1}{2}(p^2 + \omega^2 q^2)$$

$$[\hat{p}, \hat{q}] = -i\hbar$$

$$a := \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p})$$

$$a^\dagger := \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p})$$

$$\left[a, a^\dagger\right] = 1$$

Remark.

1. $a, a^\dagger$ form an algebra $\mathcal{A}$ called $*$-algebra.
2. If we only consider the operator algebra without considering H then there are a number of different ways to represent it.

   Postulate a vector $|0\rangle$ with $a\,|0\rangle$. The Hilbert space is spanned by $(a^\dagger)^n$. We can make a linear transformation to

$$b = \alpha a + \beta a^\dagger$$

$$b^\dagger = \gamma a + \delta a^\dagger$$

$\left[b, b^\dagger\right]$ if $\alpha\delta - \beta\gamma = 1$. If preserve the $*$ structure then $\delta = \alpha^*$ and $\gamma = \beta^*$

The group with such matrices

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \qquad |\alpha|^2 - |\beta|^2 = 1$$

forms the group

$$SU(1,1) := \{A \in M_2(\mathbb{C}) | A^\dagger \eta A = \eta\}$$

Then we have a different representation:$b\,|0\rangle_b = 0$. For a finite number of oscillators, the representations will be equivalent by the **Bogoliubov transformation**. But for an infinite number of oscillators, the representation can be inequivalent. Group Extensions and Group Cohomology

Definition 5.23 ((physics) partition function)**.**

$$\mathrm{Tr}_{\mathcal{H}_{single\ h.o.}} e^{-\beta H} = \frac{e^{-\frac{1}{2}\beta\omega}}{1 - e^{-\beta\omega}}$$

Suppose they have frequencies that are all a multiple of a basic harmonic which we'll denote ω_o. The motivation comes from the theory of strings.

We can write the standard sum of harmonic oscillator Hamiltonians we get, formally,

$$H^{formal} = \sum_{j=1}^{\infty} j\omega_0 (a_j^\dagger a_j + \frac{1}{2})$$

$$a_j\,|vac\rangle = 0 \qquad \forall j$$

Then the groundstate energy is infinite. However, there is a very natural way to regularize and renormalize this divergence by using the Riemann zeta function.

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{j}{2}\omega &= \frac{\omega_0}{2}\sum_{j=1}^{\infty}\frac{1}{j^{-1}} \\ &\xrightarrow{renormalize} \frac{\omega_0}{2}\zeta(-1) \\ &\overset{\text{analytic continuation}}{=} -\frac{\omega_0}{24}\end{aligned}$$

We can show that this is reasonable. In any case, things work out very nicely if we take Hamiltonian to be:

$$H = \sum_{j=1}^{\infty} j\omega_0 a_j^\dagger a_j - \frac{\omega_0}{24}$$

The dimension of the space of states of $n\omega_0$ is $p(n)$, the partitions of n.

$$(a_1^\dagger)^{l_1}(a_2^\dagger)^{l_2}\cdots(a_n^\dagger)^{l_n}\left|0\right\rangle$$

hence the vectors are in 1-1 correspondence with the conjugacy classes of S_n. This turns out to be significant in the boson-fermion correspondence in 1+1 dimensional quantum field theory.

The quantum statistical mechanical partition function of this collection of oscillators has a truly remarkable property:

Let q be a complex number with $|q| < 1$

$$\frac{1}{\Pi_{j=1}^{\infty}(1-q^j)} = (1+q+q^2+\cdots)(1+q^2+q^4+\cdots)(1+q^3+q^6+\cdots)\cdots \tag{5.1}$$

$$= 1+\sum_{n=1}^{\infty}p(n)q^n \tag{5.2}$$

(We know $\sum_{j=0}^{\infty} q^{sj} = \frac{1}{1-q^s}$ Each series in the nth bracket means how many ns are in the partitions.)

The partition function of all oscillators is:

$$Z^{osc}(\beta) = \mathrm{Tr}_{\mathcal{H}_{all\ the\ h.o.}}\, e^{-\beta H} = \frac{1}{q^{1/24}\Pi_{n=1}^{\infty}(1-q^n)}$$

Here $q = e^{-\beta\omega_0}$

Definition 5.24 (high-low temperature "duality")**.**

$$\beta^{-1/4}Z^{osc}(\beta) = \tilde{\beta}^{-1/4}Z^{osc}(\tilde{\beta})$$

$$\beta\tilde{\beta} = \left(\frac{2\pi}{\omega_0}\right)^2$$

In analytic number theory, we can rewrite $\tau = \mathrm{i}\frac{\beta\omega_0}{2\pi}$ s.t. $q = e^{-\beta\omega_0} = e^{2\pi\mathrm{i}\tau}$. We can define the Dedekind eta function:

$$\eta(\tau) := e^{2\pi\mathrm{i}\tau/24}\prod_{n=1}^{\infty}(1-q^n)$$

with crucial identity:

$$\eta(-1/\tau) = (-\mathrm{i}\tau)^{1/2}\eta(\tau)$$

Remark (Physics interpretation)**.** Consider a circular string with coordinate $\sigma \sim \sigma + 2\pi$ and the displacement of the string is $X(\sigma, t)$. The action will be:

$$S = \frac{1}{4\pi l_s^2}\int dt\int_0^{2\pi} d\sigma((\partial_t X)^2 - (\partial_\sigma X)^2)$$

Here l_s is the units of length. The solution of the classical equation of motion $(\partial_t^2 - \partial_\sigma^2)X = 0$ is :

$$X(t,\sigma) = X_0 + l_s^2 pt + \mathrm{i}\frac{l_s}{\sqrt{2}}\sum_{n=0}(\frac{\alpha_n}{n}e^{\mathrm{i}n(t+\sigma)} + \frac{\tilde{\alpha}_n}{n}e^{\mathrm{i}n(t-\sigma)})$$

with $\alpha_n = (\alpha_{-n})^*,\ \tilde{\alpha}_n = (\tilde{\alpha}_{-n})^*$

The Hamiltonian is

$$H = \frac{1}{4\pi l_s^3}\int_0^{2\pi} d\sigma((\partial_t X)^2 + (\partial_\sigma X)^2)$$

$$H = \frac{1}{2}l_s p^2 + \frac{1}{2}l_s^{-1}\sum_{n\neq 0}(\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n)$$

When we quantizing the system we find $[X_0, p] = \mathrm{i}\hbar$ and $[\alpha_n, \alpha_m] = n\delta_{n+m,0}$, $[\tilde{\alpha}_n, \tilde{\alpha}_m] = n\delta_{n+m,0}$, $[\alpha_n, \tilde{\alpha}_m] = 0$.

Then we represent thee Heisenberg algebras with vacua s.t. $\alpha_n |0\rangle = \tilde{\alpha}_n |0\rangle = 0$ for $n > 0$. By defining $a_n = \alpha_n/\sqrt{n}$ and $a_n^\dagger = \alpha_{-n}/\sqrt{n}$. Then

$$H = \frac{1}{2}l_s\hat{p}^2 + l_s^{-1}\left[\sum_{n=1}^{\infty} n(a_n^\dagger a_n) + \tilde{a}_n^\dagger\tilde{a}_n) - \frac{1}{12}\right]$$

Thus $\omega_0 = l_s^{-1}$

Definition 5.25 (Casimir energy)**.** In physics, we interpret the ground state energy with the boundary condition. The renormalization can be thought of as the comparison of Two energies: The vacuum energy with boundary and without boundary. $\sum n$ and $\int x dx$

Then we can compute the partition function of this string:

$$Z(\beta) := \mathrm{Tr}\, e^{-\beta H} = \left(\frac{R}{l_s}\right)\frac{l_s}{\sqrt{2\pi\beta l_s}}(Z^{osc}(\beta))^2 = \left(\frac{R}{l_s}\right)\frac{1}{\sqrt{2\pi\beta\omega_0}}(Z^{osc}(\beta))^2$$

Definition 5.26 (partition)**.** a partition is a sequence of nonegative integers $\lambda := \lambda_1, \lambda_2, \cdots$ s.t.

1. $\lambda_i \geq \lambda_{i+1}$
2. λ_i finally became zero.

The nonzero λ_i are called the parts of the partition.

we define $|\lambda| := \sum_i \lambda_i$. If $n = |\lambda|$ then we get a partition of n:

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$$

We say we have a partition of n with k parts.

Definition 5.27 (Young diagram)**.** We set λ_i boxes in the ith row and call the "upside-down L- shape" a Young diagram.

Definition 5.28 (multiplicity)**.** $m_i(\lambda)$ is called the multiplicity of I in λ

$$m_i(\lambda) := |\{j|\lambda_j = i\}|$$

denote the number of rows in $Y(\lambda)$ with i boxes.

Example 5.9 (Conjugate partition.)**.** If we flip $Y(\lambda)$ on the main diagonal: exchanging rows and columns. We can get another partition $Y(\lambda')$

$$\lambda'_i = |\{j|\lambda_j \geq i\}|$$

6 Centralizer Subgroups And Counting Conjugacy Classes

Definition 6.1 (centralizer subgroup)**.** Let $g \in G$, the centralizer subgroup of g (normalizer subgroup), denoted $Z(g)$

$$Z(g) := \{h = G|hg = gh\} = \{g \in G|hgh^{-1} = g\}$$

Remark.

$$Z(G) = \cap_{g\in G} Z(g)$$

We can build a 1-1 correspondence between $C(g)$ and $G/Z(g)$

Definition 6.2 (class equation)**.**

We can decompose a finite group:

$$|G| = \sum_{conj.\ classes} |C(g)|$$

because of the 1-1 correspondence

$$|C(g)| = \frac{|G|}{|Z(g)|}$$

we can get the class equation:

$$|G| = \sum_{conj.\ classes} \frac{|G|}{|Z(g)|}$$

6.1 0+1-Dimensional Gauge Theory

7 Generators And Relations

8 Representation Theory

9 The Group Of Automorphisms

10 Semidirect Products

11 Group Extensions and Group Cohomology

12 Overview of general classification theorems for finite groups

13 Categories: Groups and Groupoids

14 Lattice Gauge Theory